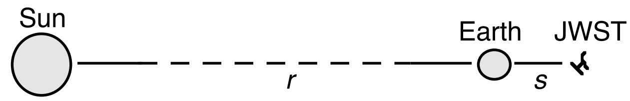
L2 Calculation

John H. Gibson, February 6, 2011

For the Earth moving in circular orbit around a fixed Sun, the Earth's centripetal acceleration is given by



$$F_{\text{Sun}} = M_{\text{Earth}} a_{C0}, \tag{1}$$

so that

$$G\frac{M_{\text{Sun}} M_{\text{Earth}}}{r^2} = M_{\text{Earth}} \omega^2 r, \tag{2}$$

or, finally

$$\omega^2 = \frac{GM_{\text{Sun}}}{r^3}. \tag{3}$$

Similarly, a satellite (e.g. the JWST) of mass m at the Earth's 2nd Lagrangian point L_2 experiences a centripetal acceleration given by

$$F_{\text{Sun}} + F_{\text{Earth}} = m a_{C2}. \tag{4}$$

If the satellite is to orbit the Sun at the same angular frequency, we must have

$$a_{C2} = \omega^2 (r+s), \tag{5}$$

so that

$$G\frac{M_{\text{Sun}} m}{(r+s)^2} + G\frac{M_{\text{Earth}} m}{s^2} = m\omega^2 (r+s), \tag{6}$$

or, after using Eq. 3 to replace ω^2 and simplifying,

$$\frac{M_{\text{Sun}}}{(r+s)^2} + \frac{M_{\text{Earth}}}{s^2} = \frac{M_{\text{Sun}}}{r^3}(r+s), \tag{7}$$

or

$$\frac{M_{\text{Sun}}}{r^2}\left(1+\frac{s}{r}\right)^{-2} + \frac{M_{\text{Earth}}}{s^2} = \frac{M_{\text{Sun}}}{r^2}\left(1+\frac{s}{r}\right). \tag{8}$$

But for $s << r$,

$$\left(1+\frac{s}{r}\right)^{-2} = 1 - 2\frac{s}{r} + \ldots, \tag{9}$$

so that

$$\frac{M_{\text{Sun}}}{r^2}\left[1 - 2\frac{s}{r} + \ldots\right] + \frac{M_{\text{Earth}}}{s^2} = \frac{M_{\text{Sun}}}{r^2}\left(1+\frac{s}{r}\right), \tag{10}$$

which simplifies to

$$\frac{M_{\text{Earth}}}{s^2} = \frac{M_{\text{Sun}}}{r^2}\left(3\frac{s}{r}\right), \tag{11}$$

or, finally

$$\boxed{s = r\left[\frac{M_{\text{Earth}}}{3M_{\text{Sun}}}\right]^{1/3}}. \tag{12}$$

But

$$\frac{M_{\text{Earth}}}{M_{\text{Sun}}} = 3.00 \times 10^{-6}, \tag{13}$$

so that

$$s = r\left[1.00 \times 10^{-6}\right]^{1/3}, \tag{14}$$

or, finally,

$$\boxed{s = 0.010\, r}. \tag{15}$$