Harmonic Motion of a Mass on a Vertical Spring

Purpose

The purpose of this experiment is to investigate harmonic motion by studying a mass oscillating on the end of a vertical spring. A sonic ranger position sensor is used to measure the position of the mass and a force sensor is used to measure the force exerted by the spring. The spring constant and frequency of oscillation are determined. In addition, the size of the mass is increased in order to introduce air resistance to the system. The damping coefficient is measured and compared to the cross-sectional area of the mass.

Background

Simple Harmonic Motion

Simple harmonic motion occurs when the restoring force on a mass is proportional to the displacement of the mass from its equilibrium position. If the restoring force is provided by a spring of force constant k and x is the displacement from equilibrium, then this relationship can be expressed as

$$F = -kx$$

which is known as Hooke's Law. The negative sign indicates the force and displacement are in opposite directions.

Application of Newton's second law yields

$$F = ma = -kx$$

$$a = \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Notice that since x isn't constant, the acceleration is not constant. The simplest method to solve this differential equation is to notice that oscillatory functions like sine and cosine have the property that if you take two derivatives then you get the same function back but multiplied by a negative constant. This suggests trying a solution of the form

$$x = x_0 + A\cos(\omega_0 t + \phi)$$

which is found to work if the frequency of oscillation is given by

$$\omega_0 = \sqrt{\frac{k}{m}}$$

In the above expression A is the amplitude of the oscillation, ϕ is the phase, and x_0 is a constant offset in case the equilibrium isn't at x=0. The relationship between the period (τ) and the angular frequency can be found by noting that the cosine function goes through one complete cycle as its argument increases by 2π. Thus, the angular frequency is given by

$$\omega_0 = \frac{2\pi}{\tau}$$

This will be helpful when making initial guesses for ω_0 when fitting the data.

Differentiating the position once gives the velocity

$$v = \frac{dx}{dt} = -\omega A\ \sin(\omega t + \phi)$$

and again gives the acceleration

$$a = \frac{dv}{dt} = -\omega^2 A\ \cos(\omega t + \phi)$$

Damped Harmonic Motion

The above discussion of simple harmonic motion neglects certain fundamental truths of the "real world." In reality, an oscillator travels through a viscous fluid, namely air, which resists the motion with a force that depends on the speed of the object. For many cases, specifically, when the oscillator mass is large and the cross-sectional area small, this resistance may be neglected with little consequence.

In cases when air resistance may not be neglected, however, there is a significant change in the behavior of the oscillator. Consider a mass-spring system subject also to an resistive force that is directly proportional to the speed

$$F_R = -bv$$

where b is a constant called the damping coefficient. The equation of motion and resulting differential equation in this case become

$$F_{net} = -kx - bv = ma$$

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

The solution to this differential equation, while rather straightforward, involves techniques not normally covered at this level so we will state the solution without proof. The nature of the solution depends upon the strength of the damping. In cases when b is small, called underdamped motion, the solution is given by

$$x = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi)$$

The "amplitude" of this oscillation decreases exponentially with a rate proportional to the damping parameter. The oscillation frequency also depends on the damping coefficient according to

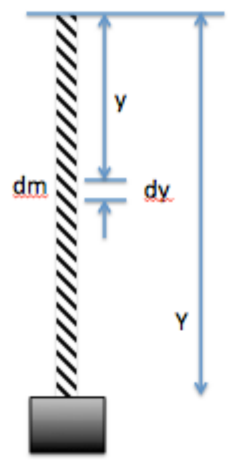
$$\omega = \sqrt{\frac{k}{m} - \frac{b}{2m}} = \sqrt{\omega_0^2 - \frac{b}{2m}}$$

where ω_0 is the undamped frequency from simple harmonic motion.

Pre-Lab Questions

1. Suppose that you have a mass oscillating on a spring and that you have measured the position of the mass and the force exerted on it by the spring. Describe a graphical technique that you could use to determine the force constant k of the spring from these data.

2. When you compare your experimentally determined frequency of oscillation (ω_0) found from the position versus time data to the theoretical value of $\sqrt{k/m}$ you will find that they do not agree very well. The primary reason is that the theoretical value was derived assuming that the spring didn't have any mass. In this question you will show that if the mass of the spring (m_s) is included that the theoretical oscillation frequency will be given by



$$\omega_0 = \sqrt{\frac{k}{m + \frac{m_s}{3}}}$$

The reason that only 1/3 of the spring mass is included in the equation is because not all of the spring is in motion. The end of the spring attached to the mass hanger moves with the same velocity v as the mass hanger. The other end of the spring is fixed and has zero velocity. Suppose that the total length of the spring at any moment is Y. Imagine dividing the spring into small pieces each of mass dm and length dy and that the distance of this piece of the spring from the fixed end is y (as shown in the figure). Show that if you assume the velocity increases linearly along the spring that the velocity of this piece will be given by

$$\text{velocity of mass segment } dm = \frac{y}{Y}v$$

Use this result to show that the kinetic energy of the spring is $\frac{1}{6}m_s v^2$. To do this, add up (integrate) the kinetic energy of each mass segment dm over the entire length of the spring. Since the kinetic energy of the spring is

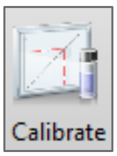
$$\frac{1}{2}\left(\frac{m_s}{3}\right)v^2$$

the effective mass of the spring that must be included in the formula for the oscillation frequency is $m_s/3$.

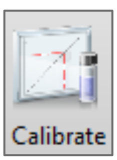
3. You probably expect that as the size of the mass increases that the amount of damping as indicated by the damping coefficient b would also increase. But what is the specific relationship between the damping coefficient and the object size? Is b proportional to the radius of the mass, or the area, or some other function? If you know b for different size objects (all of the same mass), describe a graphical method that you could use to determine the relationship between b and the size of the object.

In-Lab Procedure

1. Measure and record in your lab notebook the masses and diameters of the mass hangers that will be used in the experiment. Also measure and record the mass of your spring.
2. Arrange the iWorx MDN-100 position sensor so that it is directly below the vertical oscillation of the mass on the spring. Mount the iWorx force sensor directly above and high enough that the spring and mass can oscillate without getting too close to the position sensor.
3. Connect the myDAQ (or ELVIS workstation) to the computer with the USB cable and connect the position and force sensors to the myDAQ (or to the ELVIS). Be sure to record in your lab notebook the analog input channels you use for each sensor.
4. Open the *Physics Lab Assistant* software. Create a Position waveform using the Add button on the Analog Input Waveforms tab. Be sure to associate this waveform with the same physical channel that you connected the position sensor to previously.



5. Select the Position waveform in the Analog Input Waveforms table and use the Calibrate button to perform a two-point calibration of the sensor. For best results use two points that cover the extremes of the expected motion and then check the calibration somewhere in the middle of this range. Recall that you shouldn't place anything closer than 15 cm to the front of the position sensor.
6. Create Velocity and Acceleration waveforms using the Add button on the Derived Waveforms tab. Be sure to enter the correct relationship to compute these waveforms from previously defined waveforms.
7. Create a Force waveform using the Add button on the Analog Input Waveforms tab. Be sure to associate this waveform with the same physical channel that you connected the force sensor to previously.



8. Select the Force waveform in the Analog Input Waveforms table and use the Calibrate button to perform a two-point calibration of the sensor. Calibrate the sensor in Newtons. Use one calibration weight that is very small and the other that is twice as heavy as the mass you will use in the experiment.
9. Use the Save button under the Experiment Setup area at the top of the screen to save these waveform definitions and calibration to a file. This file can be used to restart the experiment without having to re-define the waveforms and recalibrate in the event you have to start over.
10. Mount the smallest diameter mass (just a hanger and mass without any extra sail) onto the end of the spring and hang the spring from the force sensor. Start the mass oscillating by pulling it down approximately 10 cm and releasing it. Practice doing this in such a way that the spring is not swinging side-to-side and does not have its own internal oscillation. The results improve after the mass has been oscillating for a few seconds allowing the transient oscillations to damp out.

11. Using the Acquire button, acquire a set of position and force versus time data as the mass oscillates on the spring. It may take a few practice trials to perfect your technique and get a good set of data. You want to make the mass oscillate smoothly and in a vertical line above the position sensor and obtain very smooth traces of position and force versus time.

12. Select the Waveforms tab to change the main display to show four graphs of Position, Force, Velocity, and Acceleration. By inspection of the waveform traces, identify and record the phase relationship between Position and Velocity, Velocity and Acceleration, Position and Acceleration, Force and Position, and Force and Acceleration. In other words, are the

waveforms exactly in phase with the other, are they exactly ½ cycle out of phase, or is one leading or trailing the other by ¼ cycle, etc.

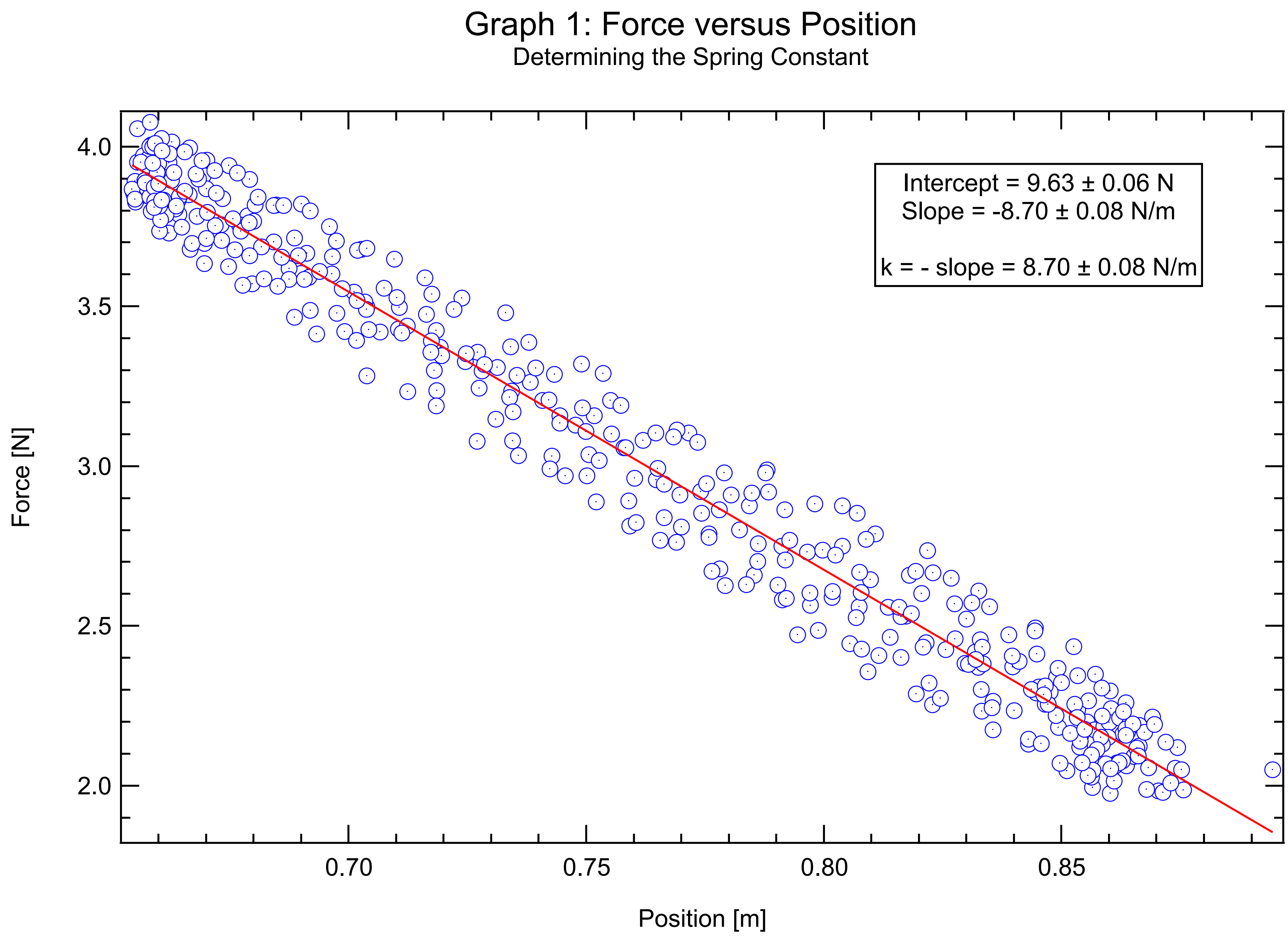
13. When you are satisfied that you have a good set of data, export the waveforms to a file. Import the data into your scientific graphing software and make plots of the force versus position and position versus time.
14. Fit the force versus position data to a straight line. Record the slope and intercept and identify the spring constant from these parameters.
15. Fit the position versus time waveform to the solution for the simple harmonic oscillator given in the introduction. Extract the frequency of oscillation, amplitude, and phase from the fit parameters and record these values on the plot and in your lab notebook.
16. Using your value for the force constant k and the mass m of the object determine a calculated value for ω_0 using the appropriate relation for simple harmonic motion. Compare this result to the experimental value from the position versus time fit. Is the agreement improved if you include the mass of the spring as described in Pre-Lab Question 2?
17. Replace the mass hanger with the next larger mass provided. Set the mass into oscillation and collect a good set of position and force data. Export the waveform data and then fit the position waveform to the solution for a damped oscillator. When performing the non-linear fit it will be necessary to make initial guesses for the fit parameters x_0, A, ω_0, ϕ, and b. Visually estimate x_0 and A from your graph. Estimate ω_0 by observing the period of oscillation. Try $\phi = 0$ rad and $b = 0.1$ kg/s as the convergence of the fit generally isn't affected by bad choices for these parameters. Extract and record the damping coefficient b and the frequency of oscillation ω.
18. Repeat the previous step for each of the masses provided. Record your results in a table with columns for the diameter of the mass, the damping coefficient, and the frequency of oscillation.

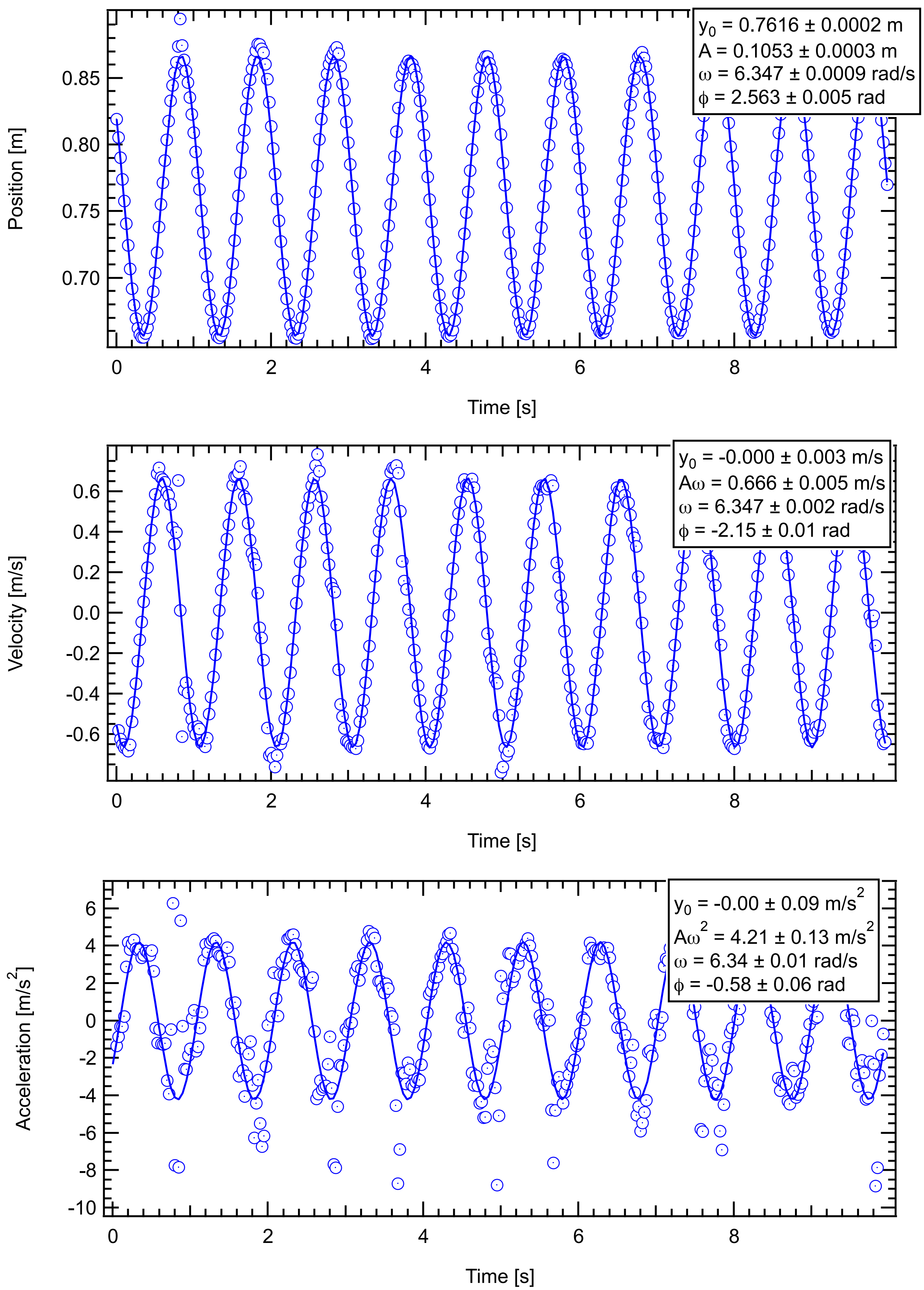
Post-Lab

1. Show how you used your plot of force versus position to determine the force constant of the spring.
2. Discuss the agreement (or lack thereof) between the value of the frequency of the simple harmonic oscillator (no damping) calculated using

$$\omega_0 = \sqrt{\frac{k}{m}}$$

   and the value obtained from the fit to the position versus time data. Was the agreement improved by including the mass of the spring as developed in Pre-Lab Question 2?
3. Discuss the dependence of the damping coefficient b on the size of the oscillating mass. Describe the dependence with a relationship and create a plot to justify your results.
4. Discuss the dependence of the angular frequency of the damped oscillator on the damping coefficient. Do your measurements follow the same trend as suggested by the theory presented in the introduction?


Graph 1: Force versus Position
Determining the Spring Constant
Intercept = 9.63 ± 0.06 N
Slope = -8.70 ± 0.08 N/m
k = - slope = 8.70 ± 0.08 N/m
Force [N]
4.0
3.5
3.0
2.5
2.0
0.70
0.75
0.80
0.85
Position [m]



y0 = 0.7616 ± 0.0002 m
A = 0.1053 ± 0.0003 m
ω = 6.347 ± 0.0009 rad/s
ϕ = 2.563 ± 0.005 rad
Position [m]
Time [s]
y0 = -0.000 ± 0.003 m/s
Aω = 0.666 ± 0.005 m/s
ω = 6.347 ± 0.002 rad/s
ϕ = -2.15 ± 0.01 rad
Velocity [m/s]
Time [s]
y0 = -0.00 ± 0.09 m/s2
Aω2 = 4.21 ± 0.13 m/s2
ω = 6.34 ± 0.01 rad/s
ϕ = -0.58 ± 0.06 rad
Acceleration [m/s2]
Time [s]

Graph 1: Position vs. Time